A Framework for Robust 2D Interface Reconstruction from Higher-Order Moments

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Motivation

Multi-fluid flows with dynamic interfaces between non-mixing fluids are fundamental to many physics simulations:

- Interaction of blast waves with structures,
- Underwater explosions,
- Bubble behavior,
- Oil-water separation,
- and many more

Accurate, robust and efficient interface reconstruction is critical.

Background

- Moment-of-fluid methods use higher order moment information for better approximation.
- Previous methods approximated the characteristic function of the interface by isocontouring a polynomial with the same moments[a].
- However, high-order polynomials oscillate heavily, making reconstruction challenging.

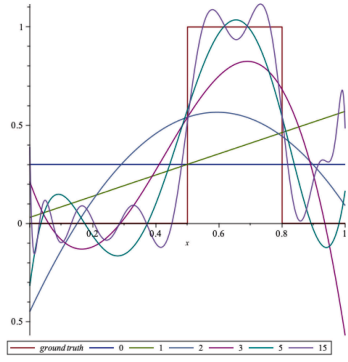
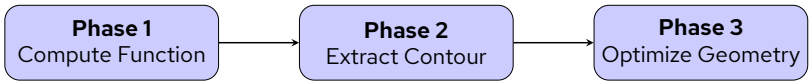
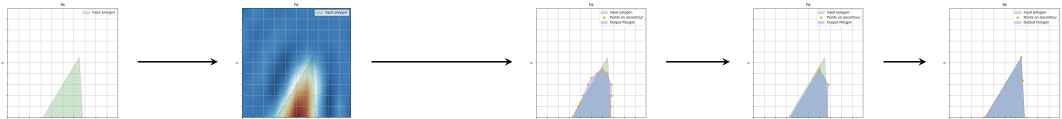


Figure: A characteristic function of reconstructed through polynomials of orders 0, 1, 2, 3, 5, and 15.

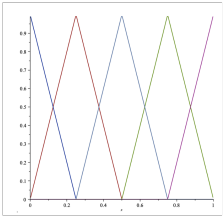
[a] Roxana Bujack and Mikhail Shashkov. "High-fidelity 2D interface reconstruction from high-order moments". In: *Computers & Fluids* 298 (2025), p. 106681. ISSN: 0045-7930. DOI: 10.1016/j.compfluid.2025.106681.

Interface Reconstruction using Finite Element Basis Functions





Compute Function



For a fixed order o, the moment $m_{p,q}$, with $p + q = o$, is:

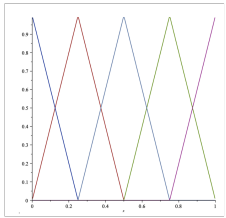
$$m_{p,q} := \int_{\mathbb{R}^2} x^p y^q f(x,y)\, dx\, dy,$$

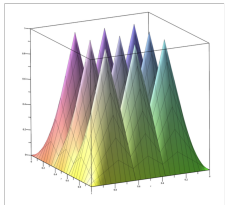
with f(x,y) being the weighted sum of 2D Finite Element basis functions.

$$f(x,y) = \sum_{i=0}^{i \leq n} \sum_{j=0}^{j \leq m} c_{i,j} g_{i,j}^{n,m}(x,y).$$

We can then construct a system of linear equations.

Compute Function

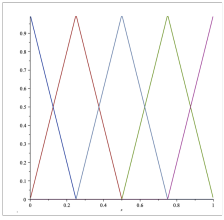




Constraining the moments to cells with a unit area, we can write the moment as:

$$
\begin{aligned}
m_{p,q} &= \int_{[0,1]^2} x^p y^q f_\chi(x,y)\,dx\,dy \\
&= \sum_{i=0}^{i\le n}\sum_{j=0}^{j\le m} c_{i,j} \int_{[0,1]^2} x^p y^q g_{i,j}^{n,m}(x,y)\,dx\,dy \\
&= \sum_{i=0}^{i\le n}\sum_{j=0}^{j\le m} c_{i,j} \int_0^1 \int_0^1 x^p y^q g_i^n(x) g_j^m(y)\,dx\,dy \\
&= \sum_{i=0}^{i\le n}\sum_{j=0}^{j\le m} c_{i,j} \int_0^1 x^p g_i^n(x)\,dx \int_0^1 y^q g_j^m(y)\,dy.
\end{aligned}
\tag{1}
$$

Compute Function



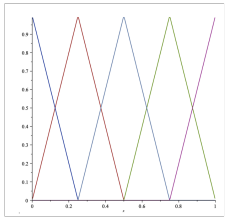
Our linear system has the form Ac=m, where $\alpha, \beta : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$ denote enumerations of the index pairs (p, q) and (i, j) (for all possible order pairs up to o), respectively. Then,

$$A_{\alpha(p,q),\beta(i,j)} = \int_0^1 x^p g_i^n(x)\, dx \int_0^1 y^q g_j^m(y)\, dy$$

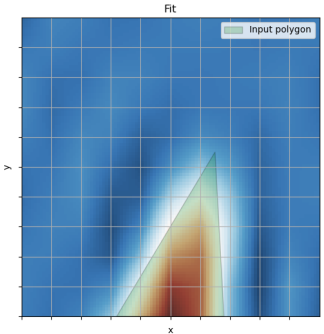
$$\mathbf{c}_{\beta(i,j)} = c_{i,j}, \quad \text{and}$$

$$\mathbf{m}_{\alpha(p,q)} = \int_{[0,1]^2} x^p y^q f(x,y)\, dx\, dy.$$

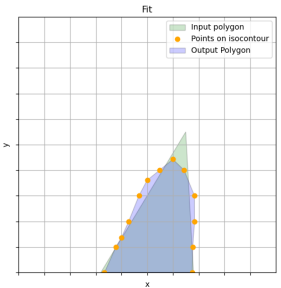
Compute Function



Evaluating the function based on the weights at every point of the mesh grid gives us a function with the same moments as the characteristic function of the interface.

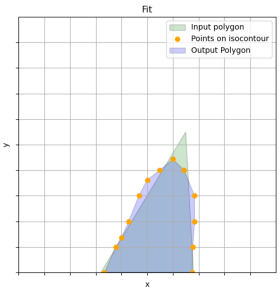


Extract Contour



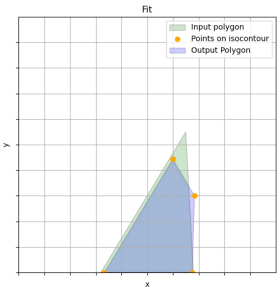
- The function converges to the characteristic function of the interface.
- The transition between high and low parts of the function will somewhere approximate the characteristic function.
- To find this transition, we 1D optimize an isovalue and extract the isocontour with Marching Squares.
- We minimize the MSE between the given and the reconstructed moments.

Optimize Geometry



- Our initial guess is not a perfect reconstruction.
- For this, we optimize the vertex positions based on the moments.
- The extracted contour is a polygon with many vertices, which we can simplify to reduce the degrees of freedom in the optimization.

Optimize Geometry



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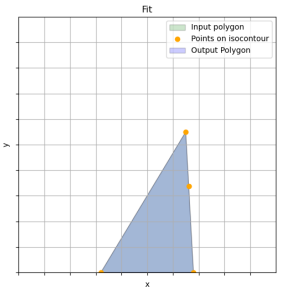

- Finally, we optimize the geometry.
- The objective is to adjust the vertex positions without altering the polygon's connectivity to minimize the difference between the polygon and given moments.
- Using PyTorch, we are tracking gradients and then optimize the geometry of the polygon by minimizing the same moment difference as before, but now with respect to the vertex positions.

Results

Our framework includes the previous state-of-the-art[1] as a special case, but includes improvements in various regards, such as:

- approximation of the characteristic function by piecewise Finite Element basis functions rather than global polynomials,
- a regularized least-squares formulation of the moment-matching system,
- incorporation of more robust iterative solvers in lieu of LU factorization,
- an optimized, instead of fixed, isovalue parameter for contour extraction, and
- additional optimization of the initial interface via advanced gradient-based optimization algorithms.

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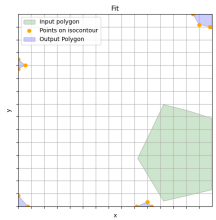
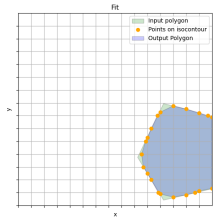
Results
Improved Solver

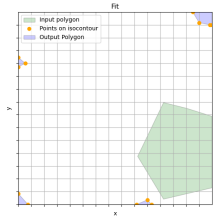


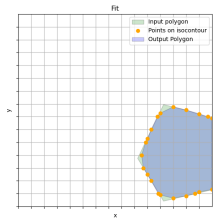


- The original implementation of the polynomial-based algorithm relies on LU factorization to solve the moment-matching linear system.

Results

Improved Solver





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- This solver becomes unreliable at higher polynomial degrees and becomes outright unusable once the degree exceeds 15 (top).

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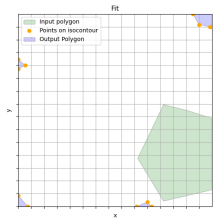
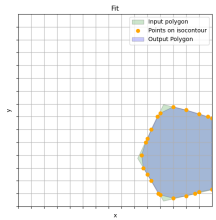
Improved Solver

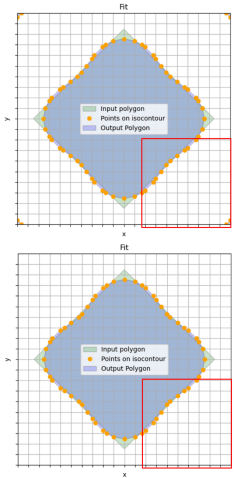




- The original implementation of the polynomial-based algorithm relies on LU factorization to solve the moment-matching linear system.
- This solver becomes unreliable at higher polynomial degrees and becomes outright unusable once the degree exceeds 15 (top).
- In contrast, solving the same linear system iteratively in a least-squares sense yields reasonable results even for polynomial degrees where LU factorization fails (bottom).

Results

Finite Element Ansatz



- Oscillations in polynomial-based approach lead to artifacts in the corners of the cell, whereas the FEM-based algorithm yields an accurate reconstruction.

Results

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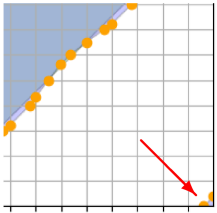

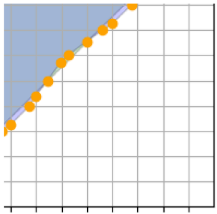

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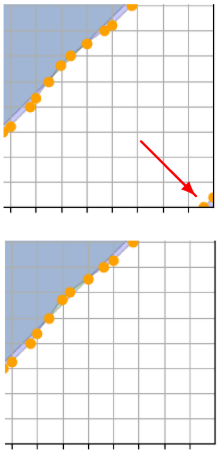

- Oscillations in polynomial-based approach lead to artifacts in the corners of the cell, whereas the FEM-based algorithm yields an accurate reconstruction.
- Simple filtering, by keeping only the persistent connected components, does not remove oscillations in this example.

Results

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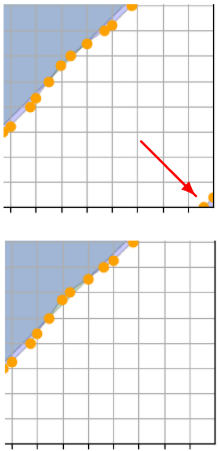
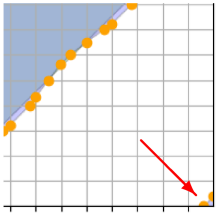
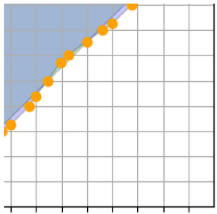


- Oscillations in polynomial-based approach lead to artifacts in the corners of the cell, whereas the FEM-based algorithm yields an accurate reconstruction.
- Simple filtering, by keeping only the persistent connected components, does not remove oscillations in this example.
- More sophisticated filtering and post-processing, e.g., comparing the results of multiple reconstructions of different orders, is needed, which would increase the complexity of the scheme.

Results

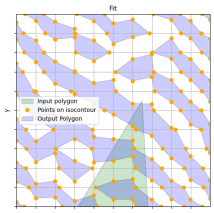
Finite Element Ansatz





- Note that while we solve both systems using the same regularized least-squares solver, this solver is more effective in controlling the oscillations of the FEM approach because of the local support of their basis functions.

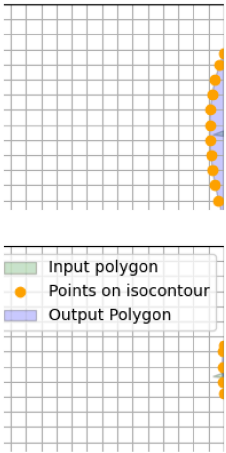
Results
Regularization



- When the finite element resolution is lower than the order of the moments, oscillations can still occur.
- Augmenting the least-squares system with a regularization term that penalizes the L2 norm of the weights can help control these oscillations.
- A maximum moment order of 15 and a FEM resolution of 10, without (top) and with (bottom) regularization.

Results

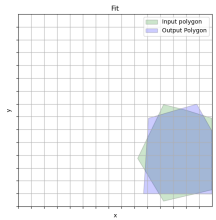
Optimized Isovalue

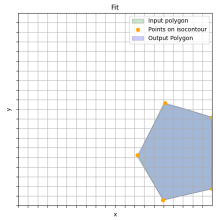


- Previously a fixed isovalue of 0.5 was used for contour extraction (top).
- Optimizing the isovalue can lead to a better initial guess, while still being computationally cheap, as it is only a 1D optimization problem (bottom).
- We seek the isovalue that minimizes the MSE between the moments of the extracted contour and the reference moments.

Results

Gradient-based Optimization

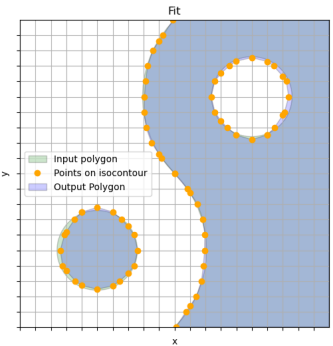




- The previous approach used direct-search optimization using the Nelder-Mead algorithm, which could fail to converge in some cases (top).
- Gradient-based optimization has shown to be more robust and can lead to better results (bottom).

Results

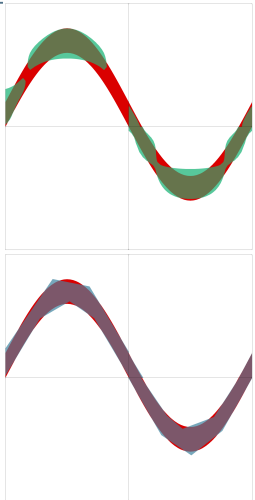
Gradient-based Optimization



- The previous approach used direct-search optimization using the Nelder-Mead algorithm, which could fail to converge in some cases.
- Gradient-based optimization has shown to be more robust and can lead to better results.
- However, optimizers tend to dominate the runtime of the algorithm.
- Omitting the decimation and optimization steps can be reasonable and in some cases even advantageous, especially in smooth interfaces.

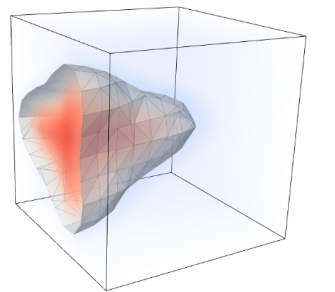
Results

Full-Mesh Example



- Full meshes highlight how cell edges interact.
- The polynomial-based approach (top) fails at this resolution and moment order, with the snake shape being reconstructed as several disconnected pieces.
- Even worse, the filtering step that should suppress oscillations actually makes the result worse by removing relevant components.
- In contrast, the FEM-based approach (bottom) can reconstruct the snake shape without any filtering and without oscillations.

Future Work



- **Extension to 3D, which is more relevant for applications, but also more challenging.**
- Application to real-world multi-fluid flow problems with real data.
- A more stable way to deal with moment noise.

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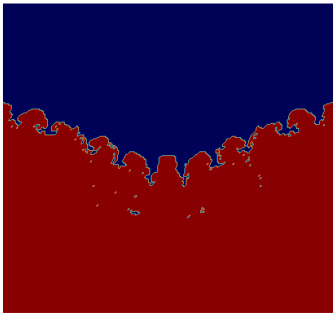

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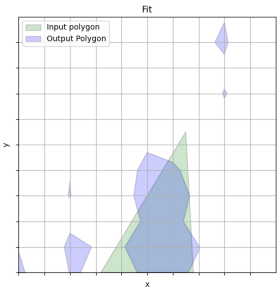

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- Application to real-world multi-fluid flow problems with real data.
- **A more stable way to deal with moment noise.**

Conclusion

- We propose a moment-of fluid 2D interface reconstruction pipeline, leveraging high-order moments within a finite element framework.
- We can generate high-quality initial guesses by extracting the isocontour of the weighted sum of finite element basis functions, which precisely reproduce the given moments.
- We can additionally optionally simplify and optimize the geometry to be as geometrically accurate as possible.

Thank you for your attention and thank you to Prof. W. Randolph Franklin for supporting travel to IMR26!